Singularities of normal affine surfaces

宮西, 正宜

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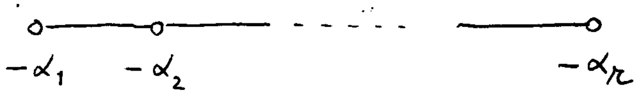
宮西正宜（阪大理）

(I) k : alg. closed field of char 0,

X : affine normal surface $/k$ containing a cylinderlike open set $U \cong U_0 \times \mathbb{A}^1_k$ (U_0 : curve); X contains such a U if $\exists\ G_a$-action on X.

d, e : positive integers s.t. $d > e$ and $(d, e) = 1$; $G = \mathbb{Z}/d\mathbb{Z} \cong \{d\text{-th roots of } 1\}$ which acts on $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ by $\zeta(x, y) = (\zeta x, \zeta^e y)$ for $\zeta \in G$; $T := \mathbb{A}^2_k/G$ with the image point Q of $(0, 0)$. $P \in X$ has <u>a cyclic quotient singularity of type</u> (d, e) $\Longleftrightarrow$ $\hat{\mathcal{O}}_{P,X} \cong \hat{\mathcal{O}}_{Q,T} \cong k[[x, y]]^G = k((x^d, \frac{y}{x^e})) \cap k[[x, y]]$. A cyclic quotient singular point P of X is a rational singular point, and if $\pi : \hat{X} \longrightarrow X$ is the minimal resolution of P, $\pi^{-1}(P)$ has the following dual graph:



,

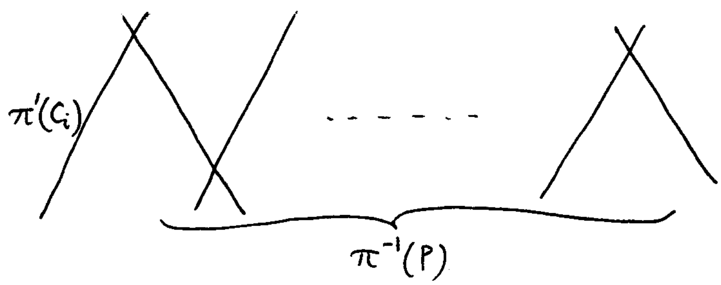
where the integers $\alpha_1, \ldots, \alpha_r$ are obtained as

$$\frac{d}{e} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_r}}} .$$

<u>Main Theorem</u>. Let X be a normal affine surface $/k$ containing a cylinderlike open set U. Then we have:

(1) Every singularity of X is a cyclic quotient singularity.

(2) $X - U = \coprod_i C_i$, disjoint union of irred. components. C_i is a rational curve with only one place at ∞ passing through at most one singular point of X; if $C_i \not\ni$ any singular point of X, $C_i \cong \mathbb{A}^1_k$; if C_i passes through a singular point P, the configuration of $\pi^{-1}(C_i)$ is:



(3) Every cyclic quotient singularity of type (d, e) is realized as a singular point of an affine normal surface containing a cylinderlike open set.

(II) <u>Application</u> <u>to</u> <u>the</u> <u>Jacobian</u> <u>Problem</u>

Recall the Jacobian Problem:

Let $\varphi : U := \mathbb{A}^2_k \longrightarrow Y := \mathbb{A}^2_k$ be an étale morphism. Is φ then an isomorphism?

Let $X :=$ normalization of Y in $k(U)$. Then

$$\begin{tikzcd} U \arrow[rr, hook, "\text{open}"] \arrow[dr, "\varphi"'] & & X \arrow[dl, "\tilde{\varphi}"] \\ & Y & \end{tikzcd}$$

where $\tilde{\varphi}$ is a finite (ramified) covering and X is an affine normal surface containing a cylinderlike open set $U \cong \mathbb{A}^1_k \times \mathbb{A}^1_k$. Thus, Main Theorem is applied to X.

Write $X - U = \coprod_i C_i$; $P_1, \ldots, P_t$: all singular points of X (if at all). If $P_j \in$ some C_i, then $C_i \subset$ the ramification locus of $\tilde{\varphi}$.

<u>Conjecture</u> 1. (i) X is nonsingular ; (ii) the branch locus of $\tilde{\varphi}$ is a union of irred. components isomorphic to $\mathbb{A}^1_k$.

To prove Conjecture (1), it suffices to affirm :

<u>Conjecture</u> 2. Let $f, g \in k[[x, y]]^G \subseteq k[[x, y]]$, where $G = \mathbb{Z}/d\mathbb{Z}$ acts on $k[[x,y]]$ via $\zeta(x,y) = (\zeta x, \zeta^e y)$. Assume that $k[[x,y]]$ is a finite $k[[f,g]]$-module. Then the ramification locus of $\mathrm{Spec}(k[[x,y]]) \to \mathrm{Spec}(k[[f,g]])$ has at least two irred. components (through the point $(0,0)$), which are not transitive to each other by the action of G.

A topological version of Conjecture 2 is :

<u>Conjecture</u> 3. Let $L(3; d, e)$ be the 3-dim$^{\underline{l}}$ lens space of type (d, e). Then there are no finite ramified coverings $f : L(3; d, e) \longrightarrow S^3$, which ramifies along a single knot.

Answer is No (cf. F. Raymond, A. Fujiki).

(III) <u>Proof</u> <u>of</u> <u>Main</u> <u>Theorem</u> (Outline)

(1) Embed $X \subset V$ (= normal projective surface)

so that V is nonsingular along $V-X$, every irred. component of $V-X$ is nonsingular, and $V-X$ has only normal crossings as singularities.

Then, the projection $p: U \longrightarrow U_0$ gives an irred. pencil Λ on V; we may assume Λ has no base points because X is affine. Hence, $\exists$ a surjective morphism $\varphi: V \longrightarrow C$ (= nonsingular complete curve) such that:

(i) $\varphi|_U = p$ with $U_0 \subset C$,

(ii) general fibers F of φ are $\cong \mathbb{P}^1_k$

(iii) $\exists$ irred. component S of $V-X$ s.t. $(S \cdot F) = 1$, i.e., S is a cross-section of φ.

(2) We may assume that U is nonsingular.

$P_1, \ldots, P_t$: all singular points of X ; ${}^\forall P_i \notin U$.

$\pi: W \rightarrow V$: minimal good resolution of $P_1, \ldots, P_t$, i.e., a) $W - \pi^{-1}\{P_1, \ldots, P_t\} \xrightarrow{\sim} V - \{P_1, \ldots, P_t\}$,

b) Every irred. component of $\pi^{-1}(P_i)$ is nonsingular,

c) $\pi^{-1}(P_i)$ has only normal crossings as singularities,

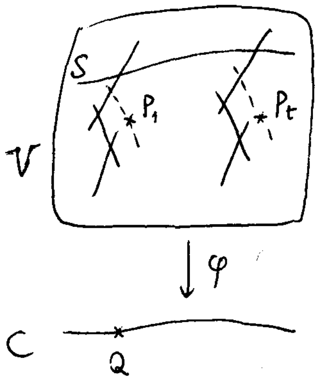
d) $\exists$ no exceptional curves (of the first kind) meeting at most two other components.

Let $L := \pi'\Lambda$ (the proper transform) and let $\psi := \varphi \cdot \pi : W \longrightarrow C$; general fibers of ψ are $\cong \mathbb{P}^1_k$. Let P be one of $P_1, \cdots, P_t$, let $F_0 := \varphi^{-1}(\varphi(P))$ and let $G_0 := \psi^{-1}(\varphi(P))$. Then $P \in F_0$ and $\pi^{-1}(P) \subset G_0$. Hence, every irred. component of $\pi^{-1}(P)$ is $\cong \mathbb{P}^1_k$, the dual graph of $\pi^{-1}(P)$ is a tree, and $\exists$ no exceptional curves of the first kind meeting 3 or more other components. Hence, $\pi^{-1}(P) \not\ni$ exceptional curves of the first kind. Let $\rho := \varphi|_X : X \longrightarrow C$ and let $Q := \rho(P)$. Then $\psi^{-1}(Q)$ is either $\mathbb{P}^1_k$ or a degenerate curve of $\mathbb{P}^1_k$. If $\psi^{-1}(Q)$ is a degenerate curve of $\mathbb{P}^1_k$, one irred. component of $\psi^{-1}(Q)$ is an exceptional curve of the first kind; after its contraction, $\psi^{-1}(Q)$ becomes $\mathbb{P}^1_k$ or a degenerate curve of $\mathbb{P}^1_k$; if it is still a degeneriate curve, we can find an exceptional curve of the first kind among irred. components. After a succession of finitely many contractions, we get $\mathbb{P}^1_k$. This implies that the dual graph of a degenerate curve of $\mathbb{P}^1_k$ contains no circular chains.

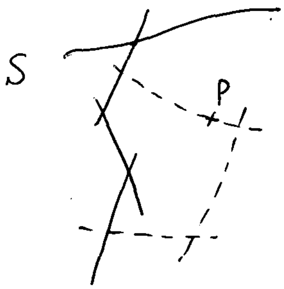
(3) <u>Claim</u> : (i) Every irred. component of $\rho^{-1}(Q)$ is a connected component of $\rho^{-1}(Q)$.

(ii) Every irred. component of $\rho^{-1}(Q)$ is a rational curve with only one place at ∞.

$\because$ $\pi^{-1}(S) \cong S$, $\pi^{-1}(S) \subset W - \pi^{-1}(X) \cong V - X$ and $(\psi^{-1}(Q) \cdot \pi^{-1}(S)) = 1$. Then $\psi^{-1}(Q) - \psi^{-1}(Q) \cap \pi^{-1}(X)$ ($\cong \varphi^{-1}(Q) - \rho^{-1}(Q)$) is connected.

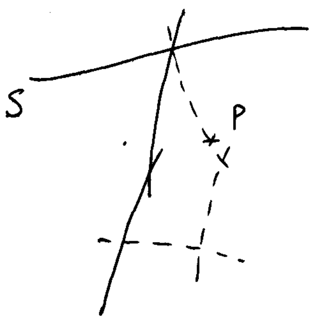


Suppose an irred. component of $\rho^{-1}(Q)$ is not a connected component. Then we have one of the configurations as shown below:



$\varphi^{-1}(Q)$ contains a cycle

or

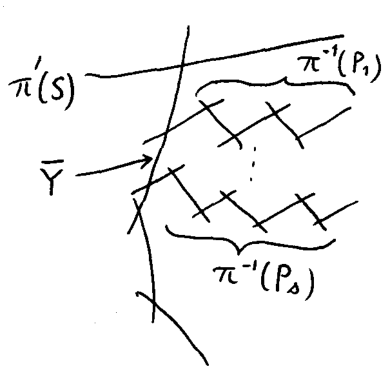


2 or more components meet S

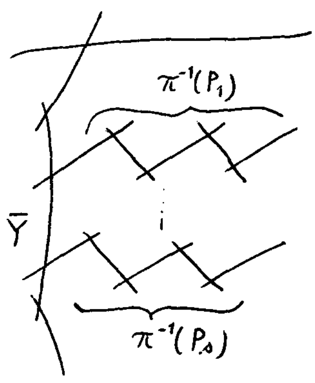
We have contradictions in both cases.

(4) <u>Claim</u>: C_i passes through at most one singular point.

⊙ Let Y be an irred. component of $\rho^{-1}(Q)$ through P. Then $\exists$ ample divisor $D > 0$ s.t. $\mathrm{Supp}(D) = V - X$ (cf. Goodman [ ]). Hence, $\exists$ ample divisor $D' > 0$ s.t. $\mathrm{Supp}(D') = (V - X) \cup (\rho^{-1}(Q) - Y)$. Replacing V by $V - \mathrm{Supp}(D')$, we may assume that $\rho^{-1}(Q) = Y$. We may assume that $\varphi^{-1}(Q) - Y$ contains no exceptional curves of the first kind. Suppose $P_1, \ldots, P_s \in Y$. Let $\bar{Y}$ be the closure of $\pi'(Y)$ in W. Then, the configuration of $\psi^{-1}(Q)$ is one of the following:



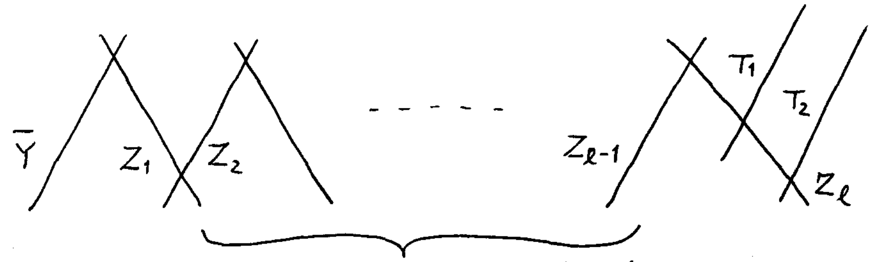
or



Note that $\bar{Y}$ is the unique exceptional curve of the first kind in $\psi^{-1}(Q)$. By contracting $\bar{Y}$, we get contradictions in both cases.

(5) <u>Claim</u> : The dual graph of $\pi^{-1}(P)$ is a linear chain.

$\because$ We may assume that $\rho^{-1}(Q) = Y$. Suppose the dual graph is not a linear chain. Then $\pi^{-1}(P)$ has the configuration :



no other components of
$\pi^{-1}(P)$ meet any one of
these curves

$\bar{Y}$ is the unique exceptional curve of the first kind in $\psi^{-1}(Q)$. Before contracting T_1 or T_2, Z_ℓ must become contractable after a succession τ of contractions. If $\tau(Z_\ell) \cap \tau(S) \neq \phi$ then $\sigma\tau(T_i) \cap \sigma\tau(S) \neq \phi$ for $i = 1, 2$, where σ is the contraction of $\tau(Z_\ell)$. If $\tau(Z_\ell) \cap \tau(S) = \phi$, $\exists$ irred. component C of $\tau(\psi^{-1}(Q))$ s.t. $C \neq \tau(Z_\ell)$, $\tau(T_i)$ $(i = 1, 2)$ and $C \cap \tau(Z_\ell) \neq \phi$. Both cases lead to contradictions.

(6) Finally, note that if $\pi^{-1}(P)$ has the linear dual

graph, P is a cyclic quotient singular point.

Q. E. D.

References

[1] J. E. Goodman, Affine open subsets of algebraic varieties and ample divisors, Ann. of Math. 89 (1969), 160-183.

[2] M. Miyanishi, Singularities of normal affine surfaces containing cylinderlike open sets, forthcoming.